

# DARBOUX TRANSFORMS AND SIMPLE FACTOR DRESSING OF CONSTANT MEAN CURVATURE SURFACES

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**ABSTRACT.** We define a transformation on harmonic maps  $N : M \rightarrow S^2$  from a Riemann surface  $M$  into the 2-sphere which depends on a parameter  $\mu \in \mathbb{C}_*$ , the so-called  $\mu$ -Darboux transformation. In the case when the harmonic map  $N$  is the Gauss map of a constant mean curvature surface  $f : M \rightarrow \mathbb{R}^3$  and  $\mu$  is real, the Darboux transformation of  $-N$  is the Gauss map of a classical Darboux transform of  $f$ . More generally, for all parameter  $\mu \in \mathbb{C}_*$  the transformation on the harmonic Gauss map of  $f$  is induced by a (generalized) Darboux transformation on  $f$ . We show that this operation on harmonic maps coincides with simple factor dressing, and thus generalize results on classical Darboux transforms of constant mean curvature surfaces [HJP97], [Bur06], [IK05]: every  $\mu$ -Darboux transform is a simple factor dressing, and vice versa.

## 1. INTRODUCTION

By the Ruh–Vilms Theorem [RV70] a constant mean curvature surface  $f : M \rightarrow \mathbb{R}^3$  of a Riemann surface  $M$  into  $\mathbb{R}^3$  is characterized by the harmonicity of its Gauss map  $N : M \rightarrow S^2$ . This fact allows to use integrable system methods for constant mean curvature surfaces: using the harmonic Gauss map one can introduce a spectral parameter  $\lambda$  to obtain the associated  $\mathbb{C}_*$ -family  $d_\lambda$  of flat connections on the trivial  $\mathbb{C}^2$  bundle over  $M$ . This family is unitary on the unit circle where it describes the associated family of harmonic maps on the universal cover  $\tilde{M}$  of  $M$ . More generally, we will use families of flat connections to construct new harmonic maps: we consider a  $\mathbb{C}_*$  family of flat connections  $d_\lambda$  so that  $d_{\lambda=1} = d$  is the trivial connection and  $d_\lambda$  satisfies a reality condition. Assuming that the map  $\lambda \rightarrow d_\lambda^{(0,1)}$  (which sends  $\lambda \in \mathbb{C}_*$  to the  $(0,1)$ -part of  $d_\lambda$ ) can be extended to a map on  $\mathbb{CP}^1$  which is meromorphic in  $\lambda$  with only a simple pole at zero we see that  $d_\lambda$  is of the form  $d_\lambda = d + (\lambda - 1)\omega^{(1,0)} + (\lambda^{-1} - 1)\omega^{(0,1)}$  with  $\omega^{(1,0)}$  and  $\omega^{(0,1)}$  of type  $(1,0)$  and  $(0,1)$  respectively. If in addition  $\omega^{(1,0)}$  is nilpotent then  $d_\lambda$  is the associated family of a harmonic map. From our point of view, dressing of a harmonic map [Uhl89, TU00] is thus the gauge of  $d_\lambda$  by an appropriate dressing matrix  $r_\lambda$ : we give conditions on  $r_\lambda$  such that  $\hat{d}_\lambda = r_\lambda \cdot d_\lambda$  is the associated family of a harmonic map. Fixing  $\mu \in \mathbb{C}_*$ , a simple factor dressing is then given by a dressing matrix  $r_\lambda$  which has a simple pole at  $\bar{\mu}^{-1}$  if  $\mu \notin S^1$ , and which depends on the choice of a  $d_\mu$ -parallel line subbundle of the trivial  $\mathbb{C}^2$  bundle over the universal cover  $\tilde{M}$  of  $M$ . We emphasize that both the associated family and the dressing are given by an operation on the harmonic map, and only the Sym–Bobenko formula [Bob91] then induces a transformation on constant mean curvature surfaces. We compare our definition of a simple factor dressing with the simple factor dressing in [DK05] which is defined on the frame of the constant mean curvature surface: indeed both transformations agree up to a rigid motion.

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In contrast to dressing, the classical Darboux transformation is originally a transformation on the level of surfaces: geometrically, two conformal immersions  $f, \hat{f} : M \rightarrow \mathbb{R}^3$  form a Darboux pair if there exists a sphere congruence enveloping both  $f$  and  $\hat{f}$ . In this case, both  $f$  and  $\hat{f}$  are isothermic. In particular, the classical Darboux transformation can be applied to a constant mean curvature surface  $f : M \rightarrow \mathbb{R}^3$ : the map  $\hat{f} : M \rightarrow \mathbb{R}^3$  is a classical Darboux transform of  $f$  if and only if  $T = \hat{f} - f$  is a solution of a certain Riccati equation. However,  $\hat{f}$  has constant mean curvature only if  $T$  additionally satisfies an initial condition.

In [BLPP08] the classical Darboux transformation is generalized to a transformation on conformal immersions  $f : M \rightarrow S^4$  by weakening the enveloping condition. It turns out that this Darboux transformation is also a key ingredient for integrable systems methods in surface theory: in the case when  $M = T^2$  is a 2-torus the spectral curve of a conformal torus  $f : T^2 \rightarrow S^2$  is essentially the set of all Darboux transforms  $\hat{f} : T^2 \rightarrow S^4$  of  $f$ . In this paper, we are interested in a (local) transformation theory for general constant mean curvature surfaces  $f : M \rightarrow \mathbb{R}^3$ , and thus have to allow the Darboux transforms  $\hat{f} : \tilde{M} \rightarrow \mathbb{R}^4$  to be defined on the universal cover  $\tilde{M}$  of  $M$ . To preserve the constant mean curvature property we will only consider so-called  $\mu$ -Darboux transforms [CLP10]: for  $\mu \in \mathbb{C}_*$  a  $\mu$ -Darboux transform  $\hat{f}$  of a constant mean curvature surface  $f$  is constructed by using a parallel section of  $d_\mu$  where  $d_\lambda$  is the associated family of the Gauss map  $N$  of  $f$ . In this case, the difference  $T = \hat{f} - f$  between  $f$  and  $\hat{f}$  also satisfies a Riccati type equation which generalizes the aforementioned equation. It turns out  $\hat{f}$  is a classical Darboux transform if and only if  $\mu \in S^1 \cup \mathbb{R}_*$ . In all cases  $T$  satisfies the required initial condition to preserve the constant mean curvature property: every  $\mu$ -Darboux transform of a constant mean curvature surface has constant mean curvature. Thus, the  $\mu$ -Darboux transformation induces a transformation on the harmonic Gauss map.

We extend this latter transformation to a  $\mu$ -Darboux transformation on harmonic maps  $N : M \rightarrow S^2$ : using the associated family of flat connections of  $N$  and a  $d_\mu$ -parallel section, we present an algebraic operation to obtain a new harmonic map on the universal cover  $\tilde{M}$  of  $M$ . It turns out, that a simple factor dressing of a harmonic map  $N$  coincides with a  $\mu$ -Darboux transform of  $N$ : the  $d_\mu$ -parallel bundle which is used to define the simple factor dressing matrix  $r_\lambda$  is spanned by the  $d_\mu$ -parallel section which gives the  $\mu$ -Darboux transform. In particular, we obtain a generalization of results on the classical Darboux transformation [HJP97, Bur06, IK05]: a  $\mu$ -Darboux transform of a constant mean curvature surface  $f : M \rightarrow \mathbb{R}^3$  is given by a simple factor dressing of the parallel constant mean curvature surface of  $f$ , and vice versa.

Many other surface classes are also linked to harmonicity, e.g., Hamiltonian stationary Lagrangians  $f : M \rightarrow \mathbb{C}^2$ . In this case, the so-called left normal  $N : M \rightarrow S^2$  of  $f$  is harmonic, and we can apply again both a simple factor dressing and the  $\mu$ -Darboux transformation on the harmonic map  $N$ . The  $\mu$ -Darboux transformation on the harmonic left normal is induced by a transformation on the level of surfaces [LR10]. In particular, though there is no Sym–Bobenko formula for Hamiltonian stationary Lagrangians, we now have an interpretation of simple factor dressing of a harmonic left normal  $N$  on the level of surfaces: the left normal of a  $\mu$ -Darboux transform of  $f$  is the simple factor dressing of  $-N$ . Moreover, in [Qui08] a dressing on (constrained) Willmore surfaces is introduced and it is shown that the simple factor dressing with simple pole at  $\mu \in \mathbb{R}_* \cup S^1$  coincides with the Darboux transformation on the conformal Gauss map of a (constrained) Willmore surface as defined in [BFL<sup>+</sup>02]. Indeed, the latter transformation can be extended [Les10]

in a way similar to what we discuss in this paper to the case when  $\mu \in \mathbb{C}_*$ , and we expect that the simple factor dressing of a (constrained) Willmore surface  $f : M \rightarrow S^4$  is the  $\mu$ -Darboux transformation on the conformal Gauss map of  $f$ .

## 2. HARMONIC MAPS AND FAMILY OF FLAT CONNECTIONS

We first recall the well-known link [Uhl89] between a harmonic map into the 2-sphere and a  $\mathbb{C}_*$ -family of flat connections. In the following we will always identify the Euclidean 4-space  $\mathbb{R}^4$  with the quaternions  $\mathbb{H}$ . In particular, we identify  $\mathbb{R}^3 = \text{Im } \mathbb{H}$  with Euclidean product  $\langle a, b \rangle = -\text{Re}(ab)$ ,  $a, b \in \text{Im } \mathbb{H}$ , and  $S^2 = \{n \in \text{Im } \mathbb{H} \mid n^2 = -1\}$ . A map  $N : M \rightarrow S^2$  from a Riemann surface  $M$  into the 2-sphere is harmonic if it is a critical point of the energy functional, that is [FLPP01], if

$$d * dN = NdN \wedge *dN.$$

Here we write for a 1-form  $\omega$

$$*\omega(X) = \omega(J_M X), \quad X \in TM,$$

where  $J_M$  denotes the complex structure of the Riemann surface  $M$ . In other words,  $*$  is the negative Hodge star operator.

We decompose  $dN$  into  $(1, 0)$  and  $(0, 1)$ -parts

$$(dN)' = \frac{1}{2}(dN - N * dN), \quad (dN)'' = \frac{1}{2}(dN + N * dN)$$

with respect to  $N$ . The harmonicity condition is now expressed by the condition that  $(dN)'$ , or equivalently  $(dN)''$ , is closed.

Every smooth map  $N : M \rightarrow S^2$  induces a quaternionic linear endomorphism  $J \in \Gamma(\text{End}(\underline{\mathbb{H}}))$  on the trivial bundle  $\underline{\mathbb{H}} = M \times \mathbb{H}$  over  $M$  by setting

$$J\phi = N\phi, \quad \phi \in \Gamma(\underline{\mathbb{H}}).$$

If we define the Hopf fields of the complex structure  $J$  (with respect to the flat connection  $d$ ) as

$$A = \frac{J(dJ) + *dJ}{4} \quad \text{and} \quad Q = \frac{J(dJ) - *dJ}{4}$$

then the closedness of  $(dN)'$  can be rephrased in terms of the complex structure  $J$ :

**Lemma 2.1.** *A smooth map  $N : M \rightarrow S^2$  is harmonic if and only if the Hopf field  $A$  of the associated complex structure  $J \in \Gamma(\text{End}(\underline{\mathbb{H}}))$  satisfies*

$$d * A = 0.$$

Note that the derivative of  $J$  is given in terms of the Hopf fields by

$$(2.1) \quad dJ = 2(*Q - *A),$$

and therefore, the condition  $d * A = 0$  is equivalent to  $d * Q = 0$ . Since

$$(2.2) \quad A = \frac{1}{2} * (dJ)' \quad \text{and} \quad Q = -\frac{1}{2} * (dJ)''$$

the Hopf fields both anti-commute with the complex structure  $J$  and have type  $(1, 0)$  and  $(0, 1)$  with respect to  $J$ , that is

$$(2.3) \quad *A = JA = -AJ, \quad *Q = -JQ = QJ.$$

In particular, by type considerations this implies

$$(2.4) \quad A \wedge Q = Q \wedge A = 0.$$

Moreover, if we decompose the trivial connection  $d$  on  $\underline{\mathbb{H}}$  into  $J$  commuting and anti-commuting parts  $d = d_+ + d_-$  then

$$(2.5) \quad d_- = A + Q$$

where we used  $d_- = \frac{1}{2}J(dJ)$  and the equations (2.1) and (2.3).

To introduce a spectral parameter  $\lambda \in \mathbb{C}_*$  we consider  $\mathbb{H}$  as a complex  $\mathbb{C}^2$  via the splitting  $\mathbb{H} = \mathbb{C} + j\mathbb{C}$  with  $\mathbb{C} = \text{span}\{1, i\}$ . In other words, if we define the complex structure  $I$  by right-multiplication by  $i \in \mathbb{H}$ , then  $\mathbb{C}^2$  can be identified with  $(\mathbb{H}, I)$ . Under this identification  $I \in \text{End}_{\mathbb{C}}(\mathbb{C}^2)$  becomes a complex linear endomorphism. For simplicity of notation we will use the same symbol for the endomorphism  $\lambda = a + Ib \in \text{End}_{\mathbb{C}}(\mathbb{C}^2)$ ,  $a, b \in \mathbb{R}$ , and the complex number  $\lambda = a + ib \in \mathbb{C}$  since  $I\phi = \phi i$  for  $\phi \in \mathbb{C}^2$ .

For  $\lambda \in \mathbb{C}_*$  we define the complex connection

$$(2.6) \quad d_\lambda = d + (\lambda - 1)A^{(1,0)} + (\lambda^{-1} - 1)A^{(0,1)}$$

on the trivial bundle  $\underline{\mathbb{C}^2} = (\underline{\mathbb{H}}, I)$  over  $M$  where

$$A^{(1,0)} = \frac{1}{2}(A - I * A) \quad \text{and} \quad A^{(0,1)} = \frac{1}{2}(A + I * A)$$

are the  $(1,0)$  and  $(0,1)$  parts of the Hopf field  $A$  with respect to the complex structure  $I$  on  $\underline{\mathbb{H}}$ . We denote by

$$\Gamma(K \text{End}_{\mathbb{C}}(\underline{\mathbb{C}^2})) = \{\omega \in \Omega^1(\text{End}_{\mathbb{C}}(\underline{\mathbb{C}^2})) \mid * \omega = I \omega\}$$

and

$$\Gamma(\bar{K} \text{End}_{\mathbb{C}}(\underline{\mathbb{C}^2})) = \{\omega \in \Omega^1(\text{End}_{\mathbb{C}}(\underline{\mathbb{C}^2})) \mid * \omega = -I \omega\}$$

the 1-forms with values in the complex linear endomorphisms of type  $(1,0)$  and  $(0,1)$ , taken with respect to the complex structure  $I$ . With this notation we have

$$A^{(1,0)} \in \Gamma(K \text{End}_{\mathbb{C}}(\underline{\mathbb{C}^2})), \quad A^{(0,1)} \in \Gamma(\bar{K} \text{End}_{\mathbb{C}}(\underline{\mathbb{C}^2})).$$

If we denote by  $E$  and  $E^\perp = Ej$  the  $\pm i$  eigenspaces of the complex structure  $J$  on  $\underline{\mathbb{H}}$  respectively then the orthogonal projections with respect to the splitting  $\underline{\mathbb{H}} = E \oplus E^\perp$  are given by

$$\pi_E = \frac{1}{2}(1 - IJ), \quad \pi_{E^\perp} = \frac{1}{2}(1 + IJ).$$

Since  $J$  is quaternionic linear  $J$  commutes with  $I$ , and so does  $A$ . Recalling (2.3) that  $A$  anti-commutes with  $J$ , we see

$$(2.7) \quad A^{(1,0)} = A\pi_{E^\perp} = \pi_E A, \quad \text{and} \quad A^{(0,1)} = A\pi_E = \pi_{E^\perp} A,$$

in particular  $(A^{(1,0)})^2 = (A^{(0,1)})^2 = 0$ , and

$$(2.8) \quad \text{im } A^{(1,0)} \subset E \subset \ker A^{(1,0)}, \quad \text{im } A^{(0,1)} \subset E^\perp \subset \ker A^{(0,1)}.$$

Since  $E^\perp = Ej$  we have  $\pi_E(\phi j) = (\pi_{E^\perp} \phi)j$  and we obtain

$$(2.9) \quad A^{(1,0)}(\phi j) = (A^{(0,1)} \phi)j, \quad \phi \in \Gamma(\underline{\mathbb{H}}).$$

Moreover,  $\lambda(\phi j) = (\bar{\lambda} \phi)j$  for  $\lambda \in \mathbb{C}$  so that the reality condition

$$(2.10) \quad d_\lambda(\phi j) = (d_{\bar{\lambda}^{-1}} \phi)j, \quad \phi \in \Gamma(\underline{\mathbb{H}}),$$

holds for the complex connection  $d_\lambda$ . In particular,  $d_\lambda$  is a quaternionic connection if and only if  $\lambda \in S^1$ .

To compute the curvature of  $d_\lambda$  we first observe that  $I$  commutes with  $J$ , and thus also with  $A$ , since  $J$  is quaternionic linear. Denoting by

$$(2.11) \quad \alpha_\lambda = (\lambda - 1)A^{(1,0)} + (\lambda^{-1} - 1)A^{(0,1)}$$

the connection form of  $d_\lambda$  we see with (2.7) that

$$\alpha_\lambda \wedge \alpha_\lambda = (2 - \lambda - \lambda^{-1})A \wedge A.$$

On the other hand, we write with (2.3)

$$(2.12) \quad A^{(1,0)} = *A \frac{J - I}{2}, \quad A^{(0,1)} = *A \frac{J + I}{2},$$

and recall (2.1) and (2.4) to obtain

$$d\alpha_\lambda = (d * A) \left( (\lambda - 1) \frac{J - I}{2} + (\lambda^{-1} - 1) \frac{J + I}{2} \right) + (\lambda + \lambda^{-1} - 2) * A \wedge * A.$$

Since  $A \wedge A = *A \wedge *A$  the curvature of  $d_\lambda$  is therefore given by

$$R_\lambda = (d * A) \left( (\lambda - 1) \frac{J - I}{2} + (\lambda^{-1} - 1) \frac{J + I}{2} \right),$$

Lemma 2.1 now yields the familiar link between harmonic maps and  $\mathbb{C}_*$ -families of flat connections:

**Theorem 2.2.** *A smooth map  $N : M \rightarrow S^2$  is harmonic if and only if the associated family of complex connections*

$$d_\lambda = d + (\lambda - 1)A^{(1,0)} + (\lambda^{-1} - 1)A^{(0,1)}$$

*on the trivial bundle  $\underline{\mathbb{C}}^2$  over  $M$  is flat for all  $\lambda \in \mathbb{C}_*$ .*

**Remark 2.3.** *Up to gauge equivalence,  $d_\lambda$  is the family of flat connections used by [Hit90] to construct the spectral curve of a harmonic torus in the 2-sphere, see [CLP10].*

The family of flat connections induces a  $S^1$ -family of harmonic maps. This family is given by  $d_\lambda$ -parallel sections and thus is only defined on the universal cover  $\tilde{M}$  of  $M$ . We denote by  $\tilde{\mathbb{H}} = \tilde{M} \times \mathbb{H}$  and  $\tilde{\mathbb{C}}^2 = \tilde{M} \times \mathbb{C}^2$  the trivial bundles over  $\tilde{M}$ .

**Theorem 2.4.** *Let  $N : M \rightarrow S^2$  be a harmonic map from a Riemann surface  $M$  into the 2-sphere,  $J$  the corresponding complex structure, and  $d_\lambda$  the associated family of flat connections. For  $\mu \in \mathbb{C}_*$  the Hopf field  $A_\mu = \frac{J(d_\mu J) + *d_\mu J}{4}$  of  $J$ , taken with respect to the flat connection  $d_\mu$ , is co-closed with respect to  $d_\mu$  for  $\mu \in \mathbb{C}_*$ , that is*

$$d_\mu * A_\mu = 0.$$

*In particular, if  $\varphi \in \Gamma(\tilde{\mathbb{H}})$  is a  $d_\mu$ -parallel section for  $\mu \in S^1$ , then  $N_\varphi = \varphi^{-1}N\varphi : \tilde{M} \rightarrow S^2$  is harmonic with respect to  $d$ . Furthermore, denoting by  $\Phi$  the endomorphism given by left multiplication by  $\varphi$ , the associated  $\mathbb{C}_*$  family of flat connections of  $N_\varphi$  is given by the gauge*

$$d_{\varphi,\lambda} = \Phi^{-1} \cdot d_{\lambda\mu}.$$

*Proof.* Write  $d_\mu = d + \alpha_\mu$  with connection form  $\alpha_\mu$  given by (2.11). The equations (2.3) and (2.12) show  $*\alpha_\mu = J\alpha_\mu = -\alpha_\mu J$  where we also used that  $[I, J] = 0$ . From type arguments we therefore obtain

$$(2.13) \quad \alpha_\mu \wedge (dJ)'' = (dJ)'' \wedge \alpha_\mu = 0.$$

Using  $d_\mu J = dJ + [\alpha_\mu, J]$  we also see with  $[I, J] = 0$

$$(2.14) \quad (d_\mu J)' = (dJ)' + [\alpha_\mu, J],$$

that is, the Hopf field of  $J$  with respect to  $d_\mu$  is

$$(2.15) \quad A_\mu = \mu A^{(1,0)} + \mu^{-1} A^{(0,1)}.$$

Since  $J$  is harmonic with respect to  $d$  and  $d\alpha_\mu + \alpha_\mu \wedge \alpha_\mu = 0$  by the flatness of  $d_\mu$ , the equations (2.14) and (2.13) give

$$d_\mu(d_\mu J)' = d((dJ)' + [\alpha_\mu, J]) + [\alpha_\mu \wedge ((dJ)' + [\alpha_\mu, J])] = 0.$$

This shows that  $*A_\mu = -\frac{1}{2}(d_\mu J)'$  is closed with respect to  $d_\mu$ . For  $\mu \in S^1$  the connection  $d_\mu$  is quaternionic, and is given by the gauge  $d_\mu = \Phi \cdot d$  of  $d$  by  $\Phi$ . Furthermore, the complex structure of  $N_\varphi$  is given by  $J_\varphi = \Phi^{-1}J\Phi$ , and we obtain  $dJ_\varphi = \text{Ad}(\Phi^{-1})(d_\mu J)$  since  $\Phi$  is parallel with respect to  $d_\mu$ . Thus,  $J_\varphi$  has Hopf field

$$A_\varphi = \Phi^{-1}A_\mu\Phi,$$

and  $0 = d_\mu(*A_\mu \circ \Phi) = \Phi \circ (d * A_\varphi)$  shows that  $N_\varphi$  is harmonic with respect to  $d$ . Finally, from (2.15) we see that

$$d_\mu + (\lambda - 1)A_\mu^{(1,0)} + (\lambda^{-1} - 1)A_\mu^{(0,1)} = d + (\lambda\mu - 1)A^{(1,0)} + ((\lambda\mu)^{-1} - 1)A^{(0,1)} = d_{\lambda\mu},$$

and gauging  $d_{\varphi,\lambda} = d + (\lambda - 1)A_\varphi^{(1,0)} + (\lambda^{-1} - 1)A_\varphi^{(0,1)}$  by  $\Phi$  we get

$$\Phi \cdot d_{\varphi,\lambda} = d_{\lambda\mu}.$$

□

**Remark 2.5.** Since  $d_\mu$  is quaternionic for  $\mu \in S^1$ , the section  $\varphi$  is unique up to a quaternionic constant and thus  $N_\varphi$  is uniquely given by  $\mu$  up to an orthogonal map. The family  $N_\varphi$  is called the associated family of  $N$ .

### 3. DRESSING OF A HARMONIC MAP INTO THE 2-SPHERE

We have seen that a harmonic map from a Riemann surface  $M$  into the 2-sphere gives rise to an associated  $\mathbb{C}_*$ -family of flat connections  $d_\lambda$  on the trivial  $\mathbb{C}^2$  bundle over  $M$  and the associated family of harmonic maps on the universal cover  $\tilde{M}$  of  $M$ . To construct new harmonic maps from given ones via dressing, one uses again the family of flat connections: Gauging  $d_\lambda$  by an appropriate  $\lambda$ -dependent dressing matrix, one can reconstruct a harmonic map from the new family of flat connections.

**Theorem 3.1.** *Let  $M$  be Riemann surface and assume that for every  $\lambda \in \mathbb{C}_*$  the connection*

$$(3.1) \quad d_\lambda = d + (\lambda - 1)\omega^{(1,0)} + (\lambda^{-1} - 1)\omega^{(0,1)}, \quad \lambda \in \mathbb{C}_*,$$

*on  $\underline{\mathbb{C}}^2$  is flat where*

$$\omega^{(1,0)} \in \Gamma(K \text{End}_{\mathbb{C}}(\underline{\mathbb{C}}^2)), \quad \omega^{(0,1)} \in \Gamma(\bar{K} \text{End}_{\mathbb{C}}(\underline{\mathbb{C}}^2))$$

are non-trivial endomorphism-valued 1-forms on the trivial bundle  $\underline{\mathbb{C}}^2 = (\underline{\mathbb{H}}, I)$  over  $M$  of type  $(1, 0)$  and  $(0, 1)$  with respect to  $I$ . If  $\omega^{(1,0)}$  is in addition nilpotent

$$(3.2) \quad (\omega^{(1,0)})^2 = 0$$

and  $\omega^{(1,0)}$  and  $\omega^{(0,1)}$  satisfy the reality condition

$$(3.3) \quad \omega^{(1,0)}(\phi j) = (\omega^{(0,1)}\phi)j, \quad \phi \in \Gamma(\underline{\mathbb{C}}^2),$$

then  $d_\lambda$  is the associated family of a harmonic map  $N : M \rightarrow S^2$ .

*Proof.* The kernel of  $\omega^{(1,0)}$  defines a line bundle  $E = \ker \omega^{(1,0)}$  away from the zeros of  $\omega^{(1,0)}$ . To show that  $E$  extends smoothly across the zeros of  $\omega^{(1,0)}$ , we equip the bundle  $K \operatorname{End}_{\mathbb{C}}(\underline{\mathbb{C}}^2)$  with a complex holomorphic structure  $D$  such that  $\omega^{(1,0)}$  is a holomorphic section in  $(K \operatorname{End}_{\mathbb{C}}(\underline{\mathbb{C}}^2), D)$ .

First note that a section  $\sigma \in \Gamma(\bar{K}K)$  can be identified with the 2-form  $\hat{\sigma} \in \Omega^2(M)$  by setting  $\hat{\sigma}(X, Y) = \sigma(X, Y) - \sigma(Y, X)$  for  $X, Y \in \Gamma(TM)$ . Under this identification, a complex holomorphic structure on the canonical bundle  $K$  is a complex linear operator  $D : \Gamma(K) \rightarrow \Omega^2(M)$  satisfying the Leibniz rule

$$D(\omega\lambda) = (D\omega)\lambda - \omega \wedge d\lambda$$

for  $\omega \in \Gamma(K)$  and  $\lambda : M \rightarrow \mathbb{C}$ . In particular, when tensoring the canonical bundle  $K$  with  $\operatorname{End}_{\mathbb{C}}(\underline{\mathbb{C}}^2)$ , we see that for every  $\eta \in \Gamma(\bar{K} \operatorname{End}_{\mathbb{C}}(\underline{\mathbb{C}}^2))$  the operator  $D : \Gamma(K \operatorname{End}_{\mathbb{C}}(\underline{\mathbb{C}}^2)) \rightarrow \Omega^2(\operatorname{End}_{\mathbb{C}}(\underline{\mathbb{C}}^2))$ ,

$$D\omega = d\omega - [\eta \wedge \omega], \quad \omega \in \Gamma(K \operatorname{End}_{\mathbb{C}}(\underline{\mathbb{C}}^2)),$$

is a complex holomorphic structure on  $K \operatorname{End}_{\mathbb{C}}(\underline{\mathbb{C}}^2)$ .

Since  $d_\lambda$  is flat for all  $\lambda \in \mathbb{C}_*$  and  $d$  is the trivial connection we have

$$0 = R_\lambda = (\lambda - 1)d\omega^{(1,0)} + (\lambda^{-1} - 1)d\omega^{(0,1)} + (\lambda^{-1} - 1)(\lambda - 1)[\omega^{(0,1)} \wedge \omega^{(1,0)}]$$

where we used that  $(\omega^{(1,0)})^2 = (\omega^{(0,1)})^2 = 0$  by (3.2) and (3.3). In particular, taking the  $\lambda$ -coefficient we have

$$0 = d\omega^{(1,0)} - [\omega^{(0,1)} \wedge \omega^{(1,0)}],$$

that is,  $\omega^{(1,0)} \in \Gamma(K \operatorname{End}_{\mathbb{C}}(\underline{\mathbb{C}}^2))$  is a holomorphic section with respect to the holomorphic structure  $D = d - \omega^{(0,1)}$ . But then  $\ker \omega^{(1,0)}$  can be extended across the zeros of  $\omega^{(1,0)}$  into a holomorphic line bundle  $E$  over  $M$ .

We now define a complex structure  $J \in \Gamma(\operatorname{End}(\underline{\mathbb{H}}))$  on  $\underline{\mathbb{H}} = E \oplus E^\perp$  by setting  $J|_E = I$  and  $J|_{E^\perp} = -I$ . Note that  $J$  is quaternionic linear since  $E^\perp = Ej$ . If we decompose  $d = d_+ + d_-$  into  $J$  commuting and anti-commuting parts, then  $d_- = \frac{1}{2}J(dJ)$ , and  $E$  and  $E^\perp$  are  $d_+$  stable, whereas  $d_-$  maps  $E$  to  $E^\perp$  and vice versa. For  $\phi \in \Gamma(E)$  we have

$$0 = (d - \omega^{(0,1)})(\omega^{(1,0)}\phi) = -\omega^{(1,0)} \wedge (d\phi - \omega^{(0,1)}\phi)$$

where we used that  $\omega^{(1,0)}$  is holomorphic with respect to  $d - \omega^{(0,1)}$  and  $E \subset \ker \omega^{(1,0)}$ . Decomposing  $d\phi = (d\phi)^{(1,0)} + (d\phi)^{(0,1)}$  into  $(1, 0)$  and  $(0, 1)$ -part with respect to  $I$ , we see by type arguments that

$$\omega^{(1,0)} \wedge (d\phi)^{(1,0)} = 0$$

and  $(d\phi)^{(0,1)} - \omega^{(0,1)}\phi$  is a 1-form with values in  $E$ . Now (3.2) shows  $\operatorname{im} \omega^{(1,0)} \subset E$  and the reality condition (3.3) gives  $\operatorname{im} \omega^{(0,1)} \subset E^\perp$ . Since  $E$  is  $d_+$  stable we thus see for  $\phi \in \Gamma(E)$  that

$$(3.4) \quad 0 = \pi_{E^\perp}((d\phi)^{(0,1)} - \omega^{(0,1)}\phi) = (\pi_{E^\perp}(d_- \phi)^{(0,1)}) - \omega^{(0,1)}\phi.$$

Recall (2.5) that  $d_- = A + Q$  and observe that

$$Q^{(0,1)}\phi = \frac{1}{2}(Q + *QI)\phi = 0$$

since  $J|_E = I$  and (2.3) holds. Substituting into (3.4) we see

$$\omega^{(0,1)} = A^{(0,1)}$$

on  $E$ . Since  $E \subset \ker \omega^{(1,0)}$  the reality condition (3.3) shows  $E^\perp \subset \ker \omega^{(0,1)}$  and from (2.8) we thus see that  $\omega^{(0,1)} = A^{(0,1)}$  on  $\underline{\mathbb{C}}^2 = E \oplus E^\perp$ . Using the reality conditions (3.3) and (2.9) we also have  $\omega^{(1,0)} = A^{(1,0)}$  on  $\underline{\mathbb{C}}^2$ . In other words,  $d_\lambda$  is the associated family of complex connections (2.6) of the map  $N : M \rightarrow S^2$  which is given by the complex structure  $J$ . In particular, since  $d_\lambda$  is flat for all  $\lambda \in \mathbb{C}_*$  the map  $N$  is harmonic by Theorem 2.2.  $\square$

**Remark 3.2.** Let  $d_\lambda$  be a family of flat connections satisfying the assumptions of Theorem 3.1. From the previous proof we see that the associated harmonic map  $N$  of  $d_\lambda$  has complex structure  $J$  with  $J|_E = I$  and  $J_{E^\perp} = -I$ . Here  $E$  is the line bundle defined by the kernel of  $\omega^{(1,0)}$ .

To obtain families of flat connections  $d_\lambda$  of the form (3.1) we observe:

**Lemma 3.3.** Let  $d_\lambda$ ,  $\lambda \in \mathbb{C}_*$ , be a family of connections on  $\underline{\mathbb{C}}^2$  satisfying

(i) the reality condition (2.10)

$$d_\lambda(\phi j) = (d_{\bar{\lambda}^{-1}}\phi)j \quad \text{for } \phi \in \Gamma(\underline{\mathbb{C}}^2),$$

(ii) the  $(0,1)$ -part of  $d_\lambda$  can be extended to a meromorphic map  $\lambda \mapsto d_\lambda^{(0,1)}$  on  $\mathbb{CP}^1$  which is holomorphic on  $\mathbb{C}_* \cup \infty$  and has a simple pole at 0, and

(iii)  $d_{\lambda=1} = d$ .

Then the family of connections  $d_\lambda$  is of the form

$$d_\lambda = d + (\lambda - 1)\omega^{(1,0)} + (\lambda^{-1} - 1)\omega^{(0,1)}$$

with  $\omega^{(1,0)} \in \Gamma(K \operatorname{End}_{\mathbb{C}}(\underline{\mathbb{C}}^2))$  and  $\omega^{(0,1)} \in \Gamma(\bar{K} \operatorname{End}_{\mathbb{C}}(\underline{\mathbb{C}}^2))$ .

*Proof.* Write  $d_\lambda = d + \omega_\lambda$  with connection form  $\omega_\lambda \in \Omega^1(\operatorname{End}_{\mathbb{C}}(\underline{\mathbb{C}}^2))$ . The conditions (ii) and (iii) imply that the  $(0,1)$ -part of  $d_\lambda$  is given by  $d^{(0,1)} + (\lambda^{-1} - 1)\omega_\lambda^{(0,1)}$  with  $\omega_\lambda^{(0,1)} \in \Gamma(\bar{K} \operatorname{End}_{\mathbb{C}}(\underline{\mathbb{C}}^2))$  for  $\lambda \in \mathbb{CP}^1$ . Now  $\lambda \mapsto \omega_\lambda^{(0,1)}$  is holomorphic on the compact  $\mathbb{CP}^1$  thus  $\omega^{(0,1)} = \omega_\lambda^{(0,1)}$  is independent of  $\lambda$ . Finally, by the reality condition (2.10) the  $(1,0)$ -part of  $d_\lambda$  is given by  $d^{(1,0)} + (\lambda - 1)\omega^{(1,0)}$  with  $\omega^{(1,0)}(\phi j) = (\omega^{(0,1)}\phi)j$  for  $\phi \in \Gamma(\underline{\mathbb{C}}^2)$ .  $\square$

By combining the previous results we can construct new harmonic maps by gauging the associated family of flat connections of a given harmonic map  $N : M \rightarrow S^2$  with an appropriate  $\lambda$ -dependent map. From this point on we will assume that  $N$  is non-trivial: if  $N$  is not a constant harmonic map, then we can assume without loss of generality, by passing to  $-N$  if necessary, that the associated family (2.6) of flat connections  $d_\lambda \neq d$  is non-trivial.



**Theorem 3.4** (Dressing). *Let  $N : M \rightarrow S^2$  be a non-trivial harmonic map from a Riemann surface  $M$  into the 2-sphere and  $d_\lambda$  the associated family (2.6) of flat connections. If  $r_\lambda : M \rightarrow \text{GL}(2, \mathbb{C})$  is a smooth map into the regular matrices with*

- (i)  $r_1 = \text{id}$  is the identity matrix for  $\lambda = 1$ ,
- (ii)  $\lambda \rightarrow r_\lambda$  is meromorphic on  $\mathbb{CP}^1$  and holomorphic at 0 and  $\infty$ ,
- (iii)  $r_\lambda$  satisfies the (generalized) reality condition

$$(r_\lambda \phi) j \sigma_\lambda = r_{\bar{\lambda}^{-1}}(\phi j), \quad \phi \in \underline{\mathbb{C}}^2,$$

with  $\sigma_\lambda \in \mathbb{C}_*$ , and

- (iv) the map  $\lambda \rightarrow \hat{d}_\lambda$  is holomorphic on  $\mathbb{C}_*$  where  $\hat{d}_\lambda = r_\lambda \cdot d_\lambda$  is the connection obtained by gauging  $d_\lambda$  by the dressing matrix  $r_\lambda$ ,

then  $\hat{d}_\lambda$  is the associated family (2.6) of a harmonic map  $\hat{N} : M \rightarrow S^2$ . The harmonic map  $\hat{N}$  is called the dressing of  $N$  by  $r_\lambda$ .

*Proof.* We first show that  $\hat{d}_\lambda$  satisfies the assumptions of Lemma 3.3. Since the reality condition (iii) gives  $r_\lambda^{-1}(\phi j) = (r_{\bar{\lambda}^{-1}}^{-1} \phi) \sigma_\lambda j$  we obtain, together with  $r_\lambda \cdot d_\lambda = r_\lambda \circ d_\lambda \circ r_\lambda^{-1}$  and the reality condition (2.10) for  $d_\lambda$ , that  $\hat{d}_\lambda(\phi j) = (\hat{d}_{\bar{\lambda}^{-1}} \phi) j$ .

From (iv) we see that  $\lambda \mapsto \hat{d}_\lambda^{(0,1)}$  is holomorphic for  $\lambda \in \mathbb{C}_*$ , and (ii) shows that this map extends holomorphically at  $\infty$  and has a simple pole at  $\lambda = 0$ . Finally  $\hat{d}_{\lambda=1} = d_{\lambda=1} = d$  by (i), and we can apply Lemma 3.3 to get

$$\hat{d}_\lambda = d + (\lambda - 1)\omega^{(1,0)} + (\lambda^{-1} - 1)\omega^{(0,1)}$$

with  $\omega^{(1,0)} \in \Gamma(K \text{End}_{\mathbb{C}}(\underline{\mathbb{C}}^2))$ ,  $\omega^{(0,1)} \in \Gamma(\bar{K} \text{End}_{\mathbb{C}}(\underline{\mathbb{C}}^2))$ . Since  $\lim_{\lambda \rightarrow \infty} \lambda^{-1} d_\lambda = A^{(1,0)}$  and  $r_\lambda$  is holomorphic at  $\infty$  we see that

$$(3.5) \quad \omega^{(1,0)} = \lim_{\lambda \rightarrow \infty} \lambda^{-1} \hat{d}_\lambda = \lim_{\lambda \rightarrow \infty} r_\lambda \cdot \lambda^{-1} d_\lambda = \text{Ad}(r_\infty) A^{(1,0)}$$

and, similarly,  $\omega^{(0,1)} = \text{Ad}(r_0) A^{(0,1)}$ . In particular, this shows  $(\omega^{(0,1)})^2 = (\omega^{(1,0)})^2 = 0$ , and since  $\hat{d}_\lambda$  satisfies the reality condition (2.10) we also have the reality condition (3.3) for the 1-forms  $\omega^{(1,0)}$  and  $\omega^{(0,1)}$ . Therefore, the family of flat connections  $\hat{d}_\lambda$  defines with Theorem 3.1 a harmonic map  $\hat{N} : M \rightarrow S^2$ .  $\square$

**Remark 3.5.** By Remark 3.2 the associated complex structure  $\hat{J}$  of the family of flat connections  $\hat{d}_\lambda$  is given by the quaternionic extension of  $\hat{J}|_{\hat{E}} = I$  where  $\hat{E}$  is given by the kernel of  $\omega^{(1,0)}$ . Equation (3.5) shows  $\hat{E} \subset \ker(\text{Ad}(r_\infty) A^{(1,0)})$ , and recalling that the  $+i$  eigenspace  $E$  of  $J$  satisfies  $E \subset \ker A^{(1,0)}$ , with equality away from the zeros of  $\ker A^{(1,0)}$ , we see

$$\hat{E} = r_\infty E.$$

Thus for  $\hat{\phi} = r_\infty \phi \in \hat{E}$  we obtain

$$\hat{J} \hat{\phi} = r_\infty \phi i = r_\infty J \phi,$$

in other words, the complex structure  $\hat{J}$  is given by extending  $\hat{J}|_{\hat{E}} = \text{Ad}(r_\infty) J|_{\hat{E}}$  quaternionically.

A particular dressing is given by prescribing that  $r_\lambda$  has only a simple pole in  $\lambda$  on  $\mathbb{C}_*$ :

**Example 3.6** (Simple factor dressing). Let  $N : M \rightarrow S^2$  be a non-trivial harmonic map from a Riemann surface into the 2-sphere with associated family  $d_\lambda$  of flat connections (2.6). Fix  $\mu \in \mathbb{C}_*$  and let  $M_\mu$  be a  $d_\mu$  parallel line subbundle of the trivial  $\tilde{\mathbb{C}}^2 = \tilde{M} \times \mathbb{C}^2$  bundle over the universal cover  $\tilde{M}$  of  $M$ . Denoting by  $\pi_\mu$  and  $\pi_\mu^\perp$  the projections on  $M_\mu$  and  $M_\mu^\perp$  with respect to the splitting  $\tilde{\mathbb{C}}^2 = M_\mu \oplus M_\mu^\perp$  we define

$$r_\lambda = \pi_\mu \circ \gamma_\lambda + \pi_\mu^\perp$$

where  $\gamma_\lambda$  is the complex linear endomorphism given by

$$\gamma_\lambda = \frac{1 - \bar{\mu}^{-1}}{1 - \mu} \frac{\lambda - \mu}{\lambda - \bar{\mu}^{-1}} \in \text{End}_{\mathbb{C}}(\mathbb{C}^2).$$

Note that  $r_\lambda = \text{id}$  for  $\mu \in S^1$  so that  $r_\lambda$  trivially satisfies the conditions of Theorem 3.4. Therefore, we will from now on assume that  $\mu \notin S^1$ . Since  $\pi_\mu(\varphi j) = (\pi_\mu^\perp \varphi)j$  and  $\bar{\gamma}_\lambda^{-1} = \gamma_{\bar{\lambda}^{-1}}$  we see that the reality condition (iii) in Theorem 3.4

$$(r_\lambda \phi)j\gamma_{\bar{\lambda}^{-1}} = r_{\bar{\lambda}^{-1}}(\phi j)$$

holds for  $\phi \in \tilde{\mathbb{C}}^2$ . As we have seen before this implies that the reality condition (2.10) holds for  $\hat{d}_\lambda = r_\lambda \cdot d_\lambda$ . Moreover,  $\lambda \mapsto r_\lambda$  is meromorphic on  $\mathbb{C}_*$  with simple zero at  $\lambda = \mu$  and simple pole at  $\lambda = \bar{\mu}^{-1}$ , and  $\lambda \mapsto r_\lambda$  is holomorphic at 0 and  $\infty$  which in particular shows (ii) of Theorem 3.4.

The condition (i) of Theorem 3.4, that is  $r_1 = \text{id}$ , trivially holds. Therefore, it only remains to verify the holomorphicity of  $\hat{d}_\lambda$ , see (iv) of Theorem 3.4. The only issue is at  $\lambda = \mu$  and  $\lambda = \bar{\mu}^{-1}$  but, since  $\hat{d}_\lambda$  satisfies the reality condition (2.10), it is enough to consider  $\lambda = \mu$ . We express  $d_\lambda$  in terms of  $d_\mu$  as

$$d_\lambda = d_\mu + (\lambda - \mu)A^{(1,0)} + (\lambda^{-1} - \mu^{-1})A^{(0,1)}$$

so that

$$\hat{d}_\lambda = r_\lambda \cdot d_\mu + (\lambda - \mu) \text{Ad}(r_\lambda)A^{(1,0)} + \frac{\mu - \lambda}{\mu\lambda} \text{Ad}(r_\lambda)A^{(0,1)}.$$

We observe that  $\text{Ad}(r_\lambda)$  has only a simple pole at  $\mu$  which shows that  $(\lambda - \mu) \text{Ad}(r_\lambda)A^{(1,0)} + \frac{\mu - \lambda}{\mu\lambda} \text{Ad}(r_\lambda)A^{(0,1)}$  is holomorphic at  $\lambda = \mu$ . Finally, we decompose  $d_\mu$  with respect to the splitting  $\tilde{\mathbb{C}}^2 = M_\mu \oplus M_\mu^\perp$

$$d_\mu = \mathcal{D} + \beta$$

into a differential operator  $\mathcal{D}$  which leaves  $M_\mu$  and  $M_\mu^\perp$  parallel, and a tensor  $\beta$  mapping  $M_\mu$  to  $M_\mu^\perp$  and vice versa. By assumption  $M_\mu$  is  $d_\mu$ -parallel, and thus  $\beta|_{M_\mu} = 0$  and  $\text{Ad}(r_\lambda)\beta = \gamma_\lambda\beta$ .

Since  $M_\mu$  and  $M_\mu^\perp$  are  $\mathcal{D}$  stable, we see  $r_\lambda \cdot \mathcal{D} = \mathcal{D}$  so that

$$\lambda \mapsto r_\lambda \cdot d_\mu = \mathcal{D} + \gamma_\lambda\beta$$

is holomorphic in  $\lambda = \mu$ . This shows that  $\lambda \mapsto \hat{d}_\lambda$  is holomorphic in  $\mathbb{C}_*$ .

Therefore,  $r_\lambda$  satisfies for all  $\mu \in \mathbb{C}_*$  the assumptions of Theorem 3.4. In particular, we obtain for every  $\mu \in \mathbb{C}_*$  and every choice of  $d_\mu$ -parallel line bundle  $M_\mu$  a harmonic map  $\hat{N} : \tilde{M} \rightarrow S^2$ , the *simple factor dressing* of  $N$  given by  $\mu$  and  $M_\mu$ . Note that for  $\mu \in S^1$  the simple factor dressing  $\hat{N} = N$  is trivial.

## 4. SIMPLE FACTOR DRESSING OF CONSTANT MEAN CURVATURE SURFACES

By the Ruh–Vilms Theorem [RV70] an immersion  $f : M \rightarrow \mathbb{R}^3$  from a Riemann surface into  $\mathbb{R}^3$  has constant mean curvature if and only if its Gauss map  $N : M \rightarrow S^2$  is harmonic. In particular, for a given constant mean curvature surface  $f$  the Gauss map  $N$  gives a harmonic complex structure  $J$  on the trivial  $\mathbb{H}$ -bundle  $\underline{\mathbb{H}}$  and we have an associated  $\mathbb{C}_*$ -family of flat complex connections (2.6)

$$d_\lambda = d + (\lambda - 1)A^{(1,0)} + (\lambda^{-1} - 1)A^{(0,1)}$$

on  $\mathbb{C}^2 = (\mathbb{H}, I)$  where  $A = \frac{1}{2}*(dJ)'$  is the Hopf field (2.2) of  $J$ . Conversely, given the family of flat connections of a harmonic map  $N : M \rightarrow S^2$  one can reconstruct a constant mean curvature surface  $f$  by the Sym–Bobenko formula. We briefly summarize this construction in our notation. Recall first that the Gauss map  $N : M \rightarrow S^2$  of a conformal immersion  $f : M \rightarrow \mathbb{R}^3$  satisfies [BFL<sup>+</sup>02]

$$*df = Ndf = -dfN.$$

The splitting of  $dN = (dN)' + (dN)''$  into  $(1, 0)$  and  $(0, 1)$ -part of  $dN$  with respect to  $N$  is the decomposition of the shape operator into trace and trace-free parts [BFL<sup>+</sup>02] so that

$$(4.1) \quad (dN)' = -Hdf$$

where  $H$  is the mean curvature of  $f$ . We will assume from now on without loss of generality that constant mean curvature surfaces have mean curvature  $H = 1$ . In this case  $A$  is the left multiplication by  $-\frac{*df}{2}$  since

$$(4.2) \quad -2 * A\phi = (dJ)'\phi = (dN)'\phi = -df\phi, \quad \phi \in \Gamma(\underline{\mathbb{H}}).$$

**Theorem 4.1** (Sym–Bobenko formula, [Bob91]). *Let  $f : M \rightarrow \mathbb{R}^3$  be a constant mean curvature surface with Gauss map  $N : M \rightarrow S^2$ . If  $d_\lambda$  is the associated family (2.6) of complex flat connections of  $N$  then  $f$  is locally given, up to translation, by*

$$(4.3) \quad f = -2 \left( \frac{\partial}{\partial t} \varphi_{e^{it}} \Big|_{t=0} \right) \varphi_{\lambda=1}^{-1}$$

where  $\varphi_\lambda \in \Gamma(\underline{\mathbb{H}})$  are  $d_\lambda$ -parallel sections on the universal cover  $\tilde{M}$  of  $M$ , depending smoothly on  $\lambda = e^{it} \in S^1$ . Conversely, every non-trivial harmonic map  $N : M \rightarrow S^2$  from a simply connected Riemann surface  $M$  into the 2-sphere gives by (4.3) a constant mean curvature surface  $f : \tilde{M} \rightarrow \mathbb{R}^3$  on the universal cover  $\tilde{M}$  of  $M$ .

*Proof.* Let  $d_\lambda$  be the associated family of flat connections of a non-trivial harmonic map  $N : M \rightarrow S^2$  and  $\varphi_\lambda \in \Gamma(\underline{\mathbb{H}})$  a smooth family of sections with  $d_\lambda \varphi_\lambda = 0$ . With  $\lambda = e^{it} \in S^1$  we obtain

$$d \left( \frac{\partial}{\partial t} \varphi_\lambda \Big|_{t=0} \right) = \frac{\partial}{\partial t} d\varphi_\lambda \Big|_{t=0} = I(A^{(0,1)} - A^{(1,0)})\varphi_1 = - * A\varphi_1.$$

Now, if  $f$  is a constant mean curvature surface then its Gauss map  $N$  has  $(dN)' \neq 0$  and  $2 * A$  is the left multiplication by  $df$ . Thus  $f$  is given, up to translation, by (4.3).

Conversely, if  $N$  is a non-trivial harmonic map then we may assume that  $(dN)' \neq 0$ , and the above computation shows that the map  $f$  defined by (4.3) satisfies  $df\varphi = 2 * A\varphi$  where  $df$  does not vanish identically. By (2.3) we have  $*df = Ndf = -dfN$  so that  $N$  is the Gauss map of the (branched) conformal immersion  $f$  and  $f$  has constant mean curvature by the Ruh–Vilms theorem.  $\square$

In particular, the associated family and the dressing of a harmonic map  $N : M \rightarrow S^2$  also induce new constant mean curvature surfaces.

**Corollary 4.2.** *Let  $f : M \rightarrow \mathbb{R}^3$  be a constant mean curvature surface and let  $d_\lambda$  be the associated family of its Gauss map  $N$ . For  $\mu \in S^1$  let  $\varphi \in \Gamma(\underline{\mathbb{H}})$  be a  $d_\mu$ -parallel section and  $N_\varphi = \varphi^{-1}N\varphi$  the associated harmonic map. If  $\varphi_\lambda$  is a smooth  $S^1$ -family of  $d_\lambda$ -parallel sections with  $\varphi_{\lambda=\mu} = \varphi$  then*

$$f_\varphi = -2\varphi^{-1} \left( \frac{\partial}{\partial t} \varphi_{e^{it}} \Big|_{t=s} \right)$$

is a constant mean curvature surface  $f_\varphi : \tilde{M} \rightarrow \mathbb{R}^3$  with Gauss map  $N_\varphi$  where  $\mu = e^{is} \in S^1$ .

*Proof.* Recall from Theorem 2.4 that  $d_{\varphi,\lambda} = \Phi^{-1} \cdot d_{\lambda\mu}$  is the associated family of  $N_\varphi$  where  $\Phi$  is the endomorphism given by left multiplication by  $\varphi$ . In particular,  $\varphi_\lambda^\mu := \varphi^{-1}\varphi_{\lambda\mu}$  is  $d_{\varphi,\lambda}$  parallel, and the Sym–Bobenko formula shows

$$f_\varphi = -2 \left( \frac{\partial}{\partial t} \varphi_{e^{it}}^\mu \Big|_{t=0} \right) (\varphi_{\lambda=1}^\mu)^{-1} = -2\varphi^{-1} \left( \frac{\partial}{\partial t} \varphi_{e^{it}} \Big|_{t=s} \right).$$

□

**Corollary 4.3.** *Let  $f : M \rightarrow \mathbb{R}^3$  be a constant mean curvature surface, and let  $N : M \rightarrow S^2$  be the Gauss map of  $f$ . Then the dressing  $\hat{N}$  of  $N$  by  $r_\lambda$  is the Gauss map of the constant mean curvature surface*

$$\hat{f} = f - 2 \left( \left( \frac{\partial}{\partial t} r_{e^{it}} \right) \Big|_{t=0} (\varphi_{\lambda=1}) \right) \varphi_{\lambda=1}^{-1},$$

where  $\varphi_\lambda \in \Gamma(\tilde{\underline{\mathbb{H}}})$  are  $d_\lambda$ -parallel sections, depending smoothly on  $\lambda = e^{it} \in S^1$ .

*Proof.* First note that  $(\frac{\partial}{\partial t} r_{e^{it}}) \Big|_{t=0}$  is in general not quaternionic linear. We recall that the associated family of  $\hat{N}$  is given by Theorem 3.4 by  $\hat{d}_\lambda = r_\lambda \cdot d_\lambda$ . Thus for a smooth  $S^1$ -family  $\varphi_\lambda$  of  $d_\lambda$ -parallel sections we see that  $\hat{\varphi}_\lambda = r_\lambda \varphi_\lambda$  is  $\hat{d}_\lambda$ -parallel. Now the Sym–Bobenko formula (4.3) and  $r_1 = \text{id}$  give the claim. □

Identifying  $\mathbb{R}^4 = \mathbb{H}$  with  $\text{gl}(2, \mathbb{C})$ -matrices of the form  $\left\{ \begin{pmatrix} a & -\bar{b} \\ b & \bar{a} \end{pmatrix} \mid a, b \in \mathbb{C} \right\}$  via

$$a_0 + ja_1 \mapsto \begin{pmatrix} a_0 & -\bar{a}_1 \\ a_1 & \bar{a}_0 \end{pmatrix}$$

the inner product in  $\mathbb{R}^3$  is given by  $\langle v, w \rangle = -\frac{1}{2} \text{tr}(vw)$  for  $v, w \in \mathbb{R}^3$ , and the coordinate frame of an immersion  $f : M \rightarrow \mathbb{R}^3$  is under this identification the unique (up to sign) smooth map  $F : \tilde{M} \rightarrow \text{SU}(2, \mathbb{C})$  with

$$e^{-\frac{u}{2}} f_x = -iF\sigma_1 F^{-1}, \quad e^{-\frac{u}{2}} f_y = -iF\sigma_2 F^{-1}, \quad N = -iF\sigma_3 F^{-1}$$

where  $z = x + iy$  is a conformal coordinate,  $e^u$  is the induced metric, and  $\sigma_l$  are the Pauli-matrices

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \text{and} \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix};$$

in particular,  $f_z = \frac{1}{2}(f_x - if_y)$  and  $f_{\bar{z}} = \frac{1}{2}(f_x + if_y)$  are given by

$$(4.4) \quad \begin{aligned} f_z &= -ie^{\frac{u}{2}} F e_- F^{-1} \\ f_{\bar{z}} &= -ie^{\frac{u}{2}} F e_+ F^{-1} \\ N &= -iF\sigma_3 F^{-1} \end{aligned}$$

with  $e_- = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$  and  $e_+ = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$ . Since the metric and the mean curvature of an immersion  $f$  are given by  $e^u = 2 \langle f_z, f_{\bar{z}} \rangle$  and  $H = 2e^{-u} \langle f_{z\bar{z}}, N \rangle$  respectively, the frame  $F$  of a constant mean curvature surface  $f$  with mean curvature  $H = 1$  and Hopf differential  $Qdz$ ,  $Q = \langle f_{zz}, N \rangle$ , satisfies with  $\langle f_z, f_z \rangle = \langle f_{\bar{z}}, f_{\bar{z}} \rangle = 0$  the equations

$$\begin{aligned} F^{-1}F_z &= \begin{pmatrix} -\frac{1}{4}u_z & Qe^{-\frac{u}{2}} \\ -\frac{1}{2}e^{\frac{u}{2}} & \frac{1}{4}u_z \end{pmatrix} \\ F^{-1}F_{\bar{z}} &= \begin{pmatrix} \frac{1}{4}u_{\bar{z}} & \frac{1}{2}e^{\frac{u}{2}} \\ -\bar{Q}e^{-\frac{u}{2}} & -\frac{1}{4}u_{\bar{z}} \end{pmatrix}. \end{aligned}$$

In particular, the Gauss–Codazzi equations for the constant mean curvature surface  $f$  are satisfied if and only if

$$d^F := d + F^{-1}dF$$

is a flat connection. Again, we can introduce the spectral parameter  $\lambda \in \mathbb{C}_*$ :

**Lemma 4.4.** *Let  $f : M \rightarrow \mathbb{R}^3$  be a constant mean curvature surface with Gauss map  $N$  and coordinate frame  $F$  and let  $F_\lambda : \tilde{M} \rightarrow \text{GL}(2, \mathbb{C})$ ,  $\lambda \in \mathbb{C}_*$ , be an extended frame of  $f$ , that is a solution of*

$$(4.5) \quad F_\lambda^{-1}(F_\lambda)_z = \begin{pmatrix} -\frac{u_z}{4} & Qe^{-\frac{u}{2}} \\ -\frac{\lambda}{2}e^{\frac{u}{2}} & \frac{u_z}{4} \end{pmatrix}$$

$$(4.6) \quad F_\lambda^{-1}(F_\lambda)_{\bar{z}} = \begin{pmatrix} \frac{u_{\bar{z}}}{4} & \frac{\lambda^{-1}}{2}e^{\frac{u}{2}} \\ -\bar{Q}e^{-\frac{u}{2}} & -\frac{u_{\bar{z}}}{4} \end{pmatrix}$$

with  $F_{\lambda=1} = F$ . Then  $F_\lambda$  gives the associated family (2.6) of flat connections of the Gauss map  $N$  of  $f$  by

$$(4.7) \quad d_\lambda = F \cdot d^{F_\lambda}$$

where  $d^{F_\lambda} = d + F_\lambda^{-1}dF_\lambda$ .

*Proof.* Recalling (4.2) that  $A$  is the left multiplication by  $-\frac{*df}{2}$  and  $d_\lambda = d + \alpha_\lambda$  with

$$\alpha_\lambda = (\lambda - 1)A^{(1,0)} + (\lambda^{-1} - 1)A^{(0,1)}$$

we see from (4.4)

$$\alpha_\lambda^{(1,0)} = -\frac{\lambda - 1}{2}e^{\frac{u}{2}}F e_- F^{-1}dz, \quad \alpha_\lambda^{(0,1)} = \frac{\lambda^{-1} - 1}{2}e^{\frac{u}{2}}F e_+ F^{-1}d\bar{z}.$$

Putting  $S_\lambda = F_\lambda F^{-1}$  we have

$$S_\lambda^{-1}dS_\lambda = F(F_\lambda^{-1}dF_\lambda - F^{-1}dF)F^{-1}$$

and using (4.5) and (4.6) we get  $S_\lambda^{-1}dS_\lambda = \alpha_\lambda$ . Thus  $d_\lambda = S_\lambda^{-1} \cdot d$  which shows the claim.  $\square$

From the previous lemma we see that for a constant  $v \in \underline{\mathbb{C}}^2 = \underline{\mathbb{H}}$  the section  $\varphi_\lambda = F F_\lambda^{-1} v$  is  $d_\lambda$ -parallel, and we obtain from (4.3)

$$f = 2 \left( \frac{\partial}{\partial t} F_{e^{it}} \Big|_{t=0} F^{-1} \right) + \text{const.}$$

Note that this coincides with the usual Sym–Bobenko formula for the extended frame: writing  $U_\lambda = F_\lambda^{-1} (F_\lambda)_z$  we see

$$\begin{aligned} \left( \frac{\partial}{\partial t} F_{e^{it}} \Big|_{t=0} F^{-1} \right)_z &= \left( \frac{\partial}{\partial t} (F_{e^{it}})_z \Big|_{t=0} \right) F^{-1} - \frac{\partial}{\partial t} F_{e^{it}} \Big|_{t=0} F^{-1} F_z F^{-1} \\ &= F \left( \frac{\partial}{\partial t} U_{e^{it}} \Big|_{t=0} \right) F^{-1} \\ &= \frac{1}{2} f_z \end{aligned}$$

where we used (4.5) and (4.4). A similar argument gives  $\left( \frac{\partial}{\partial t} F_{e^{it}} \Big|_{t=0} F^{-1} \right)_{\bar{z}} = \frac{1}{2} f_{\bar{z}}$ .

We now connect the simple factor dressing on the extended frame [TU00, DK05] with the frame independent definition in Example 3.6. We fix  $\mu \in \mathbb{C}_*$  and recall that the simple factor dressing matrix is given by

$$r_\lambda = \pi_\mu \circ \gamma_\lambda + \pi_\mu^\perp$$

where  $M_\mu$  is a  $d_\mu$ -parallel bundle,  $\pi_\mu$  and  $\pi_\mu^\perp$  denote the projections onto  $M_\mu$  and  $M_\mu^\perp$  respectively, and  $\gamma_\lambda$  is the complex linear endomorphism given by

$$\gamma_\lambda = \frac{1 - \bar{\mu}^{-1}}{1 - \mu} \frac{\lambda - \mu}{\lambda - \bar{\mu}^{-1}} \in \text{End}_{\mathbb{C}}(\mathbb{C}^2).$$

In particular, the simple factor dressing  $\hat{N}$  of  $N$  by  $r_\lambda$  has associated family of flat connections  $\hat{d}_\lambda = r_\lambda \cdot d_\lambda$  and gives by Corollary 4.3 a constant mean curvature surface  $\hat{f}$ . We denote the extended frame of  $\hat{f}$  by  $\hat{F}_\lambda$ . Then Lemma 4.4 shows that

$$\hat{d}_\lambda = \hat{F} \cdot d^{\hat{F}_\lambda}$$

with  $\hat{F} = \hat{F}_{\lambda=1}$ . Writing  $\hat{S}_\lambda = \hat{F}_\lambda \hat{F}^{-1}$ , we thus have  $r_\lambda \cdot d_\lambda = \hat{S}_\lambda^{-1} \cdot d$ , and  $d_\lambda = F \cdot d^{F_\lambda}$  gives

$$(4.8) \quad \hat{S}_\lambda = s_\lambda \circ S_\lambda \circ r_\lambda^{-1}$$

with  $S_\lambda = F_\lambda F^{-1}$ , and a  $z$ -independent  $s_\lambda$ . On the other hand, [DK05] give a simple factor dressing  $\tilde{f}$  of a constant mean curvature surface  $f$  with extended frame  $F_\lambda$ : the extended frame  $\tilde{F}_\lambda$  of  $\tilde{f}$  is given by

$$(4.9) \quad h_\lambda \circ F_\lambda = \tilde{F}_\lambda \circ g_\lambda$$

where  $h_\lambda = \pi_{M_0} \tau_\lambda + \pi_{M_0^\perp}$  is given by the choice of a constant line  $M_0 \subset \underline{\mathbb{C}}^2$  and

$$\tau_\lambda = \frac{\lambda - \mu}{\bar{\mu}(\lambda - \bar{\mu}^{-1})} \in \text{End}_{\mathbb{C}}(\mathbb{C}^2).$$

Moreover,  $g_\lambda$  is obtained from  $h_\lambda$  by replacing the constant line  $M_0$  by the line bundle  $F_\mu^{-1} M_0$  given by the extended frame, that is

$$g_\lambda = \pi_{F_\mu^{-1} M_0} \tau_\lambda + \pi_{(F_\mu^{-1} M_0)^\perp}.$$

Putting  $s_\lambda = h^{-1} h_\lambda$ ,  $h = h_{\lambda=1}$ , we get

$$s_\lambda = \pi_{M_0} \gamma_\lambda + \pi_{M_0^\perp}$$

since  $\tau_1^{-1}\tau_\lambda = \gamma_\lambda$ . By (4.7) the line  $M_\mu = FF_\mu^{-1}M_0$  is  $d_\mu$ -parallel, and the simple factor dressing matrix  $r_\lambda$  of  $M_\mu$  is given by

$$r_\lambda = Fg^{-1}g_\lambda F^{-1}.$$

By definition of  $S_\lambda = F_\lambda F^{-1}$  and  $s_\lambda = h^{-1}h_\lambda$  this shows with (4.9)

$$s_\lambda \circ S_\lambda = (h^{-1}\tilde{F}_\lambda) \circ (h^{-1}\tilde{F})^{-1} \circ r_\lambda.$$

Plugging into (4.8) we see that  $\hat{F}_\lambda = h^{-1}\tilde{F}_\lambda W$  where  $W : M \rightarrow \text{GL}(2, \mathbb{C})$  is independent of  $\lambda$ . The Sym–Bobenko formula then yields that  $\tilde{f}$  and  $\hat{f} = h^{-1}\tilde{f}h$  coincide up to translation.

## 5. DARBOUX TRANSFORMS OF HARMONIC MAPS INTO THE 2–SPHERE

The classical Darboux transformation on isothermic surfaces can be extended to a transformation on conformal maps  $f : M \rightarrow S^4$  from a Riemann surface into the 4-sphere, [BLPP08]. In the case when  $f$  is a constant mean curvature surface, one obtains a genuine generalization of the classical Darboux transformation [CLP10]. Here we consider a special case of the general Darboux transformation, the so-called  $\mu$ -Darboux transforms. These have constant mean curvature, but are only classical Darboux transforms for special spectral parameter  $\mu$ . In particular, we obtain an induced transformation on harmonic maps  $N : M \rightarrow S^2$ .

**Theorem 5.1** ([CLP10]). *Let  $f : M \rightarrow \mathbb{R}^3$  be a constant mean curvature surface in  $\mathbb{R}^3$  with Gauss map  $N$  and associated family  $d_\lambda$  of flat connections. For  $\mu \in \mathbb{C}_*$  and  $d_\mu$ -parallel section  $\varphi \in \Gamma(\tilde{\mathbb{H}})$ , define*

$$T = \frac{1}{2}(N\varphi(a-1)\varphi^{-1} + \varphi b\varphi^{-1})$$

where  $a = \frac{\mu+\mu^{-1}}{2}, b = i\frac{\mu^{-1}-\mu}{2}$ . Then  $T$  is nowhere vanishing if  $\mu \neq 1$ , and the map  $\hat{f} : \tilde{M} \rightarrow \mathbb{R}^4 = \mathbb{H}$ ,

$$\hat{f} = f + T^{-1},$$

has constant real part. Moreover,  $\text{im } \hat{f}$  is a constant mean curvature surface with Gauss map

$$\hat{N} = -T^{-1}NT.$$

The map  $\hat{f}$  is called a  $\mu$ -Darboux transform of  $f$ .

In other words, a  $\mu$ -Darboux transform is, up to a translation, a constant mean curvature surface in  $\mathbb{R}^3$ . Note that  $\hat{f}$  depends on the choice of the  $d_\mu$ -parallel section  $\varphi \in \Gamma(\tilde{\mathbb{H}})$ .

The Darboux transformation is a key ingredient [BLPP08] for integrable systems methods in surface theory. In the case when  $M = T^2$  is a 2-torus the spectral curve of a conformal torus  $f : T^2 \rightarrow S^2$  is essentially the set of all Darboux transforms  $\hat{f} : T^2 \rightarrow S^4$  of  $f$ . If  $f : T^2 \rightarrow \mathbb{R}^3$  is a constant mean curvature torus this general spectral curve is biholomorphic [CLP10] to the spectral curve of the harmonic Gauss map  $N$  of  $f$ : the spectral curve of  $N$  is given [Hit90] by the compactification of the Riemann surface which is given by the eigenlines of the holonomies of  $d_\lambda$ ,  $\lambda \in \mathbb{C}_*$ . The eigenlines are exactly given by parallel sections with multipliers, that is  $d_\mu$ -parallel sections  $\varphi$ ,  $\mu \in \mathbb{C}_*$ , of the trivial  $\mathbb{C}^2$  bundle over  $\mathbb{C}$  which satisfy  $\gamma^*\varphi = \varphi h_\gamma$  with  $h_\gamma \in \mathbb{C}_*$  for  $\gamma \in \pi_1(T^2)$ . On the other hand, parallel sections give  $\mu$ -Darboux transforms, and the multiplier condition then implies that the  $\mu$ -Darboux transform is a conformal map on the torus.

Here we are interested in (local) transformation theory of general constant mean curvature surfaces  $f : M \rightarrow \mathbb{R}^3$  and will allow the  $\mu$ -Darboux transforms to be defined on the universal cover  $\tilde{M}$  of  $M$ . Note that  $a, b \in \mathbb{C}$  in the above theorem satisfy  $a^2 + b^2 = 1$ , however,  $a, b \in \mathbb{R}$  if and only if  $\mu \in S^1$ . In this case,  $T$  is independent of the choice of the  $d_\mu$ -parallel section  $\varphi$  and  $T^{-1} = N + \frac{b}{1-a}$ . In particular,  $\hat{f} = g + \frac{b}{1-a}$  is a translate of the parallel constant mean curvature surface  $g = f + N$  of  $f$ . On the other hand, for  $\mu \in \mathbb{R}_*$  we see that  $a \in \mathbb{R}$  so that  $T$ , and thus  $\hat{f}$ , takes values in  $\mathbb{R}^3$ .

**Theorem 5.2** ([CLP10]). *Let  $f : M \rightarrow \mathbb{R}^3$  be a constant mean curvature surface. Then a constant mean curvature surface  $\hat{f} : \tilde{M} \rightarrow \mathbb{R}^4$  is a classical Darboux transform of  $f$  if and only if  $\hat{f}$  is a  $\mu$ -Darboux transform of  $f$  with  $\mu \in \mathbb{R}_* \cup S^1$ .*

By Theorem 5.1 the  $\mu$ -Darboux transformation preserves the harmonicity of the Gauss map. More generally:

**Theorem 5.3.** *Let  $N : M \rightarrow S^2$  be a non-trivial harmonic map from a Riemann surface into the 2-sphere and  $d_\lambda$  the associated family of flat connections (2.6). Define for  $\mu \in \mathbb{C}_*$  and  $d_\mu$ -parallel section  $\varphi \in \Gamma(\tilde{\mathbb{H}})$  the map  $T : \tilde{M} \rightarrow \mathbb{H}$*

$$(5.1) \quad T = \frac{1}{2}(N\varphi(a-1)\varphi^{-1} + \varphi b\varphi^{-1})$$

where  $a = \frac{\mu + \mu^{-1}}{2}$ ,  $b = i\frac{\mu^{-1} - \mu}{2}$ . Then  $T$  is nowhere vanishing if  $\mu \neq 1$ , and

$$\hat{N} = T^{-1}NT$$

is harmonic. We call  $\hat{N}$  a  $\mu$ -Darboux transform of  $N$ .

**Remark 5.4.** *Again, we emphasize that  $\hat{N}$  depends in general on the choice of the  $d_\mu$ -parallel section  $\varphi$ . However, if  $\mu \in S^1$ , then  $a, b \in \mathbb{R}$  and  $T$  is independent of  $\varphi$ . But then  $[T, N] = 0$  gives  $\hat{N} = N$  for  $\mu \in S^1$ .*

*In particular, our choice of sign for a  $\mu$ -Darboux transform is so that it coincides with the sign of the simple factor dressing of a harmonic map on  $S^1$ . However note that with this choice the  $\mu$ -Darboux transform of the Gauss map of a constant mean curvature surface  $f$  is the negative Gauss map of the  $\mu$ -Darboux transform of  $f$ .*

*Proof.* We essentially follow the proof in [CLP10] for the analogue statement for the Gauss map of a constant mean curvature surface. Putting  $\hat{z} = \varphi z \varphi^{-1}$  for  $z \in \mathbb{C}$  we write  $T = \frac{1}{2}(N(\hat{a} - 1) + \hat{b})$  and, if  $\mu \neq 1$ , then

$$(5.2) \quad 2T(1 - \hat{a})^{-1} + N = \frac{\hat{b}}{1 - \hat{a}}.$$

Since  $N^2(p) = -1$  and  $\frac{\hat{b}^2}{(1 - \hat{a})^2} = \frac{1 + \hat{a}}{1 - \hat{a}} \neq -1$  for all  $\mu \in \mathbb{C}_*, \mu \neq 1$ , this shows that  $T(p) \neq 0$  for all  $p \in M$ . Next we observe with (2.6) and (2.12) that

$$(5.3) \quad d_\mu = d + *A(J(a-1) + b)$$

which shows with (2.2) that

$$0 = d_\mu \varphi = d\varphi - (dN)'T\varphi$$

and thus  $d\hat{z} = [(dN)'T, \hat{z}]$  for  $z \in \mathbb{C}$ . Differentiating (5.1) gives with  $\hat{a}^2 + \hat{b}^2 = 1$  the Riccati type equation

$$(5.4) \quad dT = (dN)''\frac{\hat{a} - 1}{2} - T(dN)'T,$$



which shows

$$NdT - *dT = -((dN)''(N(\hat{a} - 1) + (NT + TN)(dN)'T) .$$

Since  $d\hat{N} = [\hat{N}, T^{-1}dT] + T^{-1}dNT$  we thus obtain

$$d\hat{N} + \hat{N}d * \hat{N} = \frac{1}{2}T^{-1}(dN + N * dN)(\hat{b} - (\hat{a} - 1)\hat{N}) .$$

Now (5.1) gives  $-(\hat{a} - 1) + T\hat{N}(\hat{a} - 1) - T\hat{b} = 0$ , that is

$$\hat{N} = T^{-1} + \frac{\hat{b}}{\hat{a} - 1} ,$$

and using the Riccati type equation (5.4) we obtain

$$d * \hat{Q} = d * A$$

for the Hopf fields of  $\hat{N}$  and  $N$ . This shows that  $\hat{N}$  is harmonic.  $\square$

Note that for  $\mu \in \mathbb{R}_*$  the equation (5.4) is independent of the choice of the parallel section  $\varphi$ . In particular, if  $N$  is the Gauss map of a constant mean curvature surface  $f : M \rightarrow \mathbb{R}^3$  then the solutions of the Riccati equation (5.4) give [BFL<sup>+</sup>02] the classical Darboux transforms of  $f$ . The condition (5.2) then guarantees that  $\hat{f} = f + T^{-1}$  has constant mean curvature.

We can now generalize the results on  $\mu$ -Darboux transforms for constant mean curvature surfaces [CLP10] and Hamiltonian stationary Lagrangians in [LR10]: for a conformal immersion  $f : M \rightarrow \mathbb{R}^4$  from a Riemann surface  $M$  into 4-space, the Gauss map  $\nu : M \rightarrow \text{Gr}_2(\mathbb{R}^4)$  is a map from  $M$  into the Grassmannian of 2-planes in  $\mathbb{R}^4$ . Identifying  $\text{Gr}_2(\mathbb{R}^4) = S^2 \times S^2$  the Gauss map  $\nu$  gives rise to two maps  $N, R : M \rightarrow S^2$  satisfying

$$*df = Ndf = -dfR .$$

$N$  and  $R$  are called the *left* and *right normal* of  $f$ . From [BFL<sup>+</sup>02] we know that the  $(1, 0)$ -part of  $dN$  with respect to  $N$  is given by  $(dN)' = -dfH$  for some quaternion valued function  $H : M \rightarrow \mathbb{H}$  which satisfies  $RH = HN$ .

Examples of surfaces with harmonic left normal are constant mean curvature surfaces in 3-space, minimal surfaces in 3-space, or Hamiltonian stationary Lagrangian immersions in  $\mathbb{C}^2 = \mathbb{R}^4$ . All surfaces with harmonic left normal are constrained Willmore [LR10]. If a surface  $f : M \rightarrow \mathbb{R}^4$  has harmonic left normal we can associate again a family of flat connections  $d_\lambda = d + (\lambda - 1)A^{(1,0)} + (\lambda^{-1} - 1)A^{(0,1)}$  on  $\underline{\mathbb{H}} = \underline{\mathbb{C}}^2$  where  $A$  is the Hopf field of the associated complex structure  $J$  of  $N$ . Note that the family  $d_\lambda$  is trivial if and only if  $f$  is a minimal surface.

**Theorem 5.5.** *Let  $f : M \rightarrow \mathbb{R}^4$  be a conformal immersion with harmonic left normal  $N : M \rightarrow S^2$  which is not a minimal surface, so that  $(dN)' = -dfH$  with non-trivial  $H : M \rightarrow \mathbb{H}$ . For  $\mu \in \mathbb{C}_*$  let  $\varphi \in \Gamma(\underline{\mathbb{H}})$  be a  $d_\mu$ -parallel section of the associated family of flat connections of  $N$ . For  $\mu \neq 1$  put  $T = \frac{1}{2}(N\varphi(a - 1)\varphi^{-1} + \varphi b\varphi^{-1})$  with  $a = \frac{\mu + \mu^{-1}}{2}$ ,  $b = i\frac{\mu^{-1} - \mu}{2}$ , and*

$$\hat{f} = f + (HT)^{-1}$$

*away from the (isolated) zeros of  $H$ .*

*Then the map  $\hat{f}$  is either constant, or a (branched) conformal immersion with harmonic left normal  $\hat{N} = -T^{-1}NT$ .*

*Proof.* As before (5.3) we have  $d_\mu = d + *A(J(a-1) + b)$ , so that with  $(dN)' = -dfH$  for a  $d_\mu$ -parallel section  $\varphi \in \Gamma(\widetilde{\mathbb{H}})$

$$d\varphi = -dfHT\varphi.$$

Putting  $\beta = HT\varphi$  this gives  $0 = df \wedge d\beta$  which implies  $*d\beta = -Rd\beta$  by type arguments. In particular,  $d\beta$  has only isolated zeros if  $\beta$  is not constant [FLPP01]. From Theorem 5.3 we see that  $T$  has no zeros so that  $(HT)^{-1}$  is defined away from the zeros of  $H$ , and

$$d\hat{f} = df + d(HT)^{-1} = df + d(\varphi\beta^{-1}) = -(HT)^{-1}d\beta\varphi^{-1}(HT)^{-1}$$

shows that  $\hat{f}$  is either constant, or a branched conformal immersion with

$$*d\hat{f} = -(HT)^{-1}R(HT)d\hat{f}.$$

Using  $RH = HN$  we see that in the latter case  $\hat{f}$  has left normal

$$\hat{N} = -T^{-1}NT.$$

Theorem 5.3 therefore shows that the left normal  $\hat{N}$  of  $\hat{f}$  is harmonic.  $\square$

**Remark 5.6.** Since  $d_\mu$ -parallel sections are holomorphic, the arguments in [LR10] for the special case of Hamiltonian stationary Lagrangians show that  $\hat{f}$  as defined in the above theorem is a generalized Darboux transform of  $f$ . We call  $\hat{f}$  a  $\mu$ -Darboux transform of  $f$  as it arises from a  $d_\mu$ -parallel section  $\varphi$  for  $\mu \in \mathbb{C}_*$ .

Similarly, a  $\mu$ -Darboux transformation is defined on the conformal Gauss map of a (constrained) Willmore surface  $f : M \rightarrow S^4$  and an analogue of Theorem 5.3 holds [Les10].

## 6. DARBOUX TRANSFORMATION AND SIMPLE FACTOR DRESSING

We show that the  $\mu$ -Darboux transformation and the simple factor dressing of a harmonic map coincide. In particular, a  $\mu$ -Darboux transform of a constant mean curvature surface  $f : M \rightarrow \mathbb{R}^3$  is given by a simple factor dressing of the Gauss map of the parallel surface  $g$  of  $f$ , and vice versa. This generalizes results for classical Darboux transformations [HJP97], [Bur06], [IK05]. Moreover, since the  $\mu$ -Darboux transformation is defined for all surfaces  $f : M \rightarrow \mathbb{R}^4$  with harmonic left normal, the simple factor dressing on the harmonic left normal can thus also be given an interpretation on the level of surfaces.

**Theorem 6.1.** *Let  $N : M \rightarrow S^2$  be a non-trivial harmonic map. Then every  $\mu$ -Darboux transform of  $N$  is given by a simple factor dressing, and vice versa.*

*More precisely, if we denote by  $d_\lambda$  the associated family of flat connections of  $N$  and put  $M_\mu = \varphi\mathbb{C}$  for a  $d_\mu$ -parallel section  $\varphi \in \Gamma(\widetilde{\mathbb{H}})$ ,  $\mu \in \mathbb{C}_*$ , then the simple factor dressing  $\hat{N}$  of  $N$  with respect to  $M_\mu$  is the  $\mu$ -Darboux transform of  $N$  with respect to  $\varphi$ , that is*

$$\hat{N} = T^{-1}NT$$

*with  $T = \frac{1}{2}(N\varphi(a-1)\varphi^{-1} + \varphi b\varphi^{-1})$  and  $a = \frac{\mu+\mu^{-1}}{2}$ ,  $b = i\frac{\mu^{-1}-\mu}{2}$ .*

*Proof.* In this proof we adapt the arguments in [Qui08] to the case of harmonic maps  $N : M \rightarrow S^2$ , and generalize her setting from  $\mu \in \mathbb{R}_* \cup S^1$  to the general case  $\mu \in \mathbb{C}_*$ . As before, we denote by  $\hat{z} = \varphi z \varphi^{-1}$  for  $z \in \mathbb{C}$ , and recall that  $\hat{a}^2 + \hat{b}^2 = 1$ . Let  $J$  be the complex structure of  $N$  and  $E$  the  $+i$  eigenspace of  $J$ . Putting  $\rho = \frac{1-a}{2}$  and  $\hat{T} = T\hat{\rho}^{-1}$  we first show that the  $+i$  eigenspace of the complex structure  $\hat{J}$  of  $\hat{N} = T^{-1}NT$  is given by  $\hat{E} = \hat{T}E$ : the equation (5.2) shows  $(\hat{T} + N)^2 = -1 + \hat{\rho}^{-1}$ , that is,

$$\hat{T}^2 + \hat{T}N + N\hat{T} = \hat{\rho}^{-1}.$$

From this we see that  $N$  commutes with

$$\hat{\rho}T^{-2} = 1 + N\hat{T}^{-1} + \hat{T}^{-1}N$$

and thus  $[T^2\hat{\rho}^{-1}, N] = 0$ . For  $\hat{\phi} = \hat{T}\phi$ ,  $\phi \in E$  we therefore obtain

$$\hat{N}\hat{\phi} = \hat{T}N\phi = \hat{\phi}i,$$

and  $\hat{E}$  is the  $+i$  eigenspace of  $\hat{J}$ .

Since  $\hat{N}$  is completely determined by the  $+i$  eigenspace of  $\hat{J}$  it is enough to show by Remark 3.5 that  $\hat{E} = r_\infty E$  where  $r_\lambda = \pi_\mu \circ \gamma_\lambda + \pi_\mu^\perp$ . Here  $\pi_\mu$  and  $\pi_\mu^\perp$  are the projections onto  $M_\mu$  and  $M_\mu^\perp$  respectively, and  $\gamma_\lambda = \frac{1-\bar{\mu}^{-1}}{1-\mu} \frac{\lambda-\mu}{\lambda-\bar{\mu}^{-1}}$ . We first observe that  $a-1 = \frac{\mu^{-1}(\mu-1)^2}{2}$  and  $b = i\frac{\mu^{-1}(1-\mu^2)}{2}$  so that

$$\frac{b}{1-a} = i\frac{\mu+1}{\mu-1}.$$

Since  $\varphi \in \Gamma(M_\mu)$  we have  $r_\infty\varphi = \varphi\frac{1-\bar{\mu}^{-1}}{1-\mu}$  and  $r_\infty(\varphi j) = \varphi j$ , so that (5.2) shows

$$(\hat{T} + N - I)\varphi = \varphi\frac{2i}{\mu-1} = -(r_\infty)\frac{2I}{1-\bar{\mu}^{-1}}\varphi$$

and

$$(\hat{T} + N - I)(\varphi j) = -\varphi j\frac{2i}{1-\bar{\mu}^{-1}} = -(r_\infty)\frac{2I}{1-\bar{\mu}^{-1}}\varphi j,$$

in other words,

$$\hat{T} + N - I = -r_\infty \circ \frac{2I}{1-\bar{\mu}^{-1}}.$$

Finally, for  $\phi \in E$  we have  $(N - I)\phi = 0$  since  $E$  is the  $+i$  eigenspace of  $J$ , and thus

$$\hat{T}\phi = -r_\infty\phi\frac{2i}{1-\bar{\mu}^{-1}}.$$

This proves that  $\hat{T}E = r_\infty E$ , and thus  $\hat{N}$  is the simple factor dressing of  $N$  by  $r_\lambda$ .

□

As an immediate consequence of Theorem 6.1 and Theorem 5.1 simple factor dressing and the  $\mu$ -Darboux transformation are essentially the same for constant mean curvature surfaces:

**Theorem 6.2.** *The Gauss map of a  $\mu$ -Darboux transform  $\hat{f}$  of a constant mean curvature surface  $f : M \rightarrow \mathbb{R}^3$  is a simple factor dressing of the Gauss map of the parallel surface  $g = f + N$  of  $f$ , and vice versa.*

More generally, if  $f$  is a surface with harmonic left normal  $N$ , then Theorem 6.1 and Theorem 5.5 show that a simple factor dressing of  $N$  is induced by a transformation on the surface  $f$ :

**Theorem 6.3.** *Let  $f : M \rightarrow \mathbb{R}^4$  be a conformal immersion with harmonic left normal  $N : M \rightarrow S^2$  which is not a minimal surface. Then a simple factor dressing of  $-N$  is the left normal of a  $\mu$ -Darboux transform of  $f$ , and vice versa.*

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